MATH1010H: WEEK 2

ABSTRACT. In this week, we study the *limit of sequences*. The topics will include *limit of sequences*, *monotone convergence theorem*, *Sandwich theorem*, *arithmetic of limit*.

1. LIMIT OF A SEQUENCE

A sequence is a list of numbers, that is

$$a_1, a_2, a_3, \ldots.$$

We denote it as $(a_n)_{n=1}^{\infty}$ or (a_n). Sometimes use a_n as well.

- We can consider the sequence as a function $f : \mathbb{N} \to Y$, that $f(n) = a_n$.

Example 1.1. *Sequence appears everywhere!!*

- *Let* $a_n = \frac{1}{n}$. $\quad a_n$ *is getting closer and closer to* 0
- *Let* $a_n = (-1)^n \frac{1}{n}$. $\quad a_n$ *is getting closer and closer to* 0.
- *Let* $a_n = 1 + (-1)^n \frac{1}{n}$.
- *Let* $a_1 = 0.9, a_2 = 0.99, a_3 = 0.999, \ldots$.
- *Let* $b_n = (-1)^n$, *that is* $-1, 1, -1, 1, -1, \ldots$
- *Temperature sequence* $t_1, t_2, t_3, \ldots$

Definition 1.2. *Let* (a_n) *be a sequence of real numbers. If* n *is getting larger and larger, the value* a_n *is getting closer and closer to some* $L \in \mathbb{R}$, *then we say that* L *is the limit of the sequence and we denoted it by*

$$\lim_{n\to\infty} a_n = L.$$

We could also write $a_n \to L$. *In this case, we say that* (a_n) *is convergent, or* (a_n) *converges to* L. *Otherwise, we say that* (a_n) *is divergent.*

- A number L is the limit of the sequence (a_n) if the numbers a_n become closer and closer to L.

Example 1.3.

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

$$\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0.$$

$$\lim_{n\to\infty}(1+(-1)^n\frac{1}{n}) = 1.$$

$$\lim_{n\to\infty}(-1)^n \qquad \textit{does not exists!!}$$

Problems:
1. Given (a_n), is the sequence (a_n) convergent or divergent?
For instance: Does the sequence $a_n = \sqrt{n+1} - \sqrt{n}$ convergent?
2. If a_n is convergent then determine the value of its limit.

Example 1.4. *Prove that* $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

Answer: We will often use the following trick. For any $a, b > 0$ we have

$$a - b = \frac{(a-b)(a+b)}{a+b} = \frac{a^2 - b^2}{a+b}.$$

Applying to $\sqrt{n+1} - \sqrt{n}$, we obtain

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{\sqrt{n+1}^2 - \sqrt{n}^2}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}}.\end{aligned}$$

As n is getting larger and larger, the value $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is getting closer and closer to 0, we obtain the desired result.

Definition 1.5 (Bounded sequence)**.** *We say that* (a_n) *is bounded if there exists a positive number* M *such that*

$$|a_n| \leqslant M, \quad \forall n \in \mathbb{N}.$$

Theorem 1.6 (Monotone convergence theorem)**.** *Let* a_n *be a monotone sequence of real numbers* ($a_n \leqslant a_{n+1}$ *or* $a_n \geqslant a_{n+1}$) *and the sequence is bounded, then the sequence is convergent.*

Example 1.7. *Let* $a_1 = 2$ *and* $a_{n+1} = \sqrt{a_n + 2}$. *Prove that* $a_n \geqslant a_{n+1}$ *for all* $n \in \mathbb{N}$.

Mathematical induction, is a technique for proving a statement, a formula or an inequality is true for each natural number.

Example 1.8. *Prove that* $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
Answer: *First step: It is hold for* $n = 1$.

Second step: Assume the statement is true for the value $n = k$, that is

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

We intend to show the statement is true for $n = k + 1$. Applying the assumption, we arrive

$$1 + 2 + \ldots k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

By mathematical induction, the statement is true for each $n \in \mathbb{N}$.

Answer of Example 1.7. First step: Proves that the statement is true for the initial value. $a_1 \geqslant a_2$.

Second step: Assume the statement is true for the value k (that is $a_k \geqslant a_{k+1}$). We intend to show that the statement is true for $k+1$ (that is $a_{k+1} \geqslant a_{k+2}$). Observe that

$$\begin{aligned} a_{k+1} - a_{k+2} &= \sqrt{a_k + 2} - \sqrt{a_{k+1} + 2} \\ &= \frac{a_k - a_{k+1}}{\sqrt{a_k + 2} + \sqrt{a_{k+1} + 2}} \geqslant 0, \end{aligned}$$

we obtain that $a_{k+1} \geqslant a_{k+2}$.

So, by mathematical induction, it is proved that $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$.

Corollary 1.9. *Combining Theorem 1.6 with Example 1.7, we conclude that the sequence (a_n) of 1.7 is convergent. (Can you guess the limit of this sequence?)*

Let $L = \lim_{n\to\infty} a_n$. Then

$$L = \sqrt{L+2}.$$

Squaring the identity we obtain $L^2 = L + 2$, and this gives that $L = 2$ or $L = -1$. Since $a_n \geqslant 0$ for all $n \in \mathbb{N}$, we obtain $L = 2$.

Theorem 1.10 (A famous constant)**.** *Let $a_n = (1 + \frac{1}{n})^n$ then the sequence (a_n) is convergent.*

Since, (the proof is given at the end of this note),

- $a_n \leqslant a_{n+1}$.
- $a_n \leqslant 3$,

applying monotone convergence theorem we obtain the result. Denote this limit by

$$e := \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

Now we come to a different method for finding the limit of a given sequence.

Example 1.11. *Find*

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n}}+\frac{n^2}{n^2+10}).$$

Algebraic properties of limits.

Theorem 1.12. *Let* (a_n) *and* (b_n) *be sequence of real numbers. Suppose that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$. *Then*

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L+M;$$

$$\lim_{n\to\infty}(a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L-M.$$

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty}(b_n) = LM;$$

Moreover if $M \neq 0$,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}.$$

Example 1.13. *Find* $\lim_{n\to\infty}(2+\frac{1}{n})$.

- Let $a_n = c$ then $\lim_{n\to\infty} a_n = c$. Also write it as $\lim_{n\to\infty} c = c$.

Example 1.14. *Find* $\lim_{n\to\infty}\frac{n^2+3}{5n^2+7}$.

Example 1.15. *Find*

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n}}+\frac{n^2}{n^2+10}).$$

Answer: Since

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0,$$

and

$$\lim_{n\to\infty}\frac{n^2}{n^2+10} = 1,$$

we arrive

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n}}+\frac{n^2}{n^2+10}) = \lim_{n\to\infty}\frac{1}{\sqrt{n}} + \lim_{n\to\infty}\frac{n^2}{n^2+10} = 1.$$

Theorem 1.16 (Sandwich or squeeze theorem)**.** *Let* $(a_n), (b_n)$ *and* (c_n) *be sequence of real numbers. If* $a_n \leqslant b_n \leqslant c_n$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

Example 1.17. *Find* $\lim_{n\to\infty} \frac{\sin n}{n}$.

Answer: Note that
$$-\frac{1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n}$$
and
$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$
By sandwich theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Example 1.18. *Find* $\lim_{n\to\infty} \frac{\sin n^2 + \sqrt{7}\cos n}{n}$.

Answer: Note that
$$-\frac{10}{n} \leqslant \frac{\sin n^2 + \sqrt{7}\cos n}{n} \leqslant \frac{10}{n},$$
and
$$\lim_{n\to\infty} -\frac{10}{n} = \lim_{n\to\infty} \frac{10}{n} = 0.$$
By sandwich theorem, $\lim_{n\to\infty} \frac{\sin n^2 + \sqrt{7}\cos n}{n} = 0$.

Example 1.19. *Let* $a_n = \frac{n}{n^2+1} + \ldots + \frac{n}{n^2+n}$. *Find* $\lim_{n\to\infty} a_n$.

Answer: For $1 \leqslant i \leqslant n$ we have
$$\frac{n}{n^2+n} \leqslant \frac{n}{n+i} \leqslant \frac{n}{n^2+1},$$
and hence
$$\frac{n^2}{n^2+n} \leqslant a_n \leqslant \frac{n^2}{n^2+1}.$$
Since
$$\lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1,$$
by sandwich theorem, $\lim_{n\to\infty} a_n = 1$.

Now we show that the sequence $a_n = (1+\frac{1}{n})^n$ is convergent. Note that
$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{n(n-1)\ldots(n-k+1)}{k!} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{k-1}{n}).
\end{aligned}$$

Similarly we have

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+\sum_{k=1}^{n+1}\frac{1}{k!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\dots(1-\frac{k-1}{n+1}).$$

For each $1 \leqslant i \leqslant n-1$ we have

$$1-\frac{i}{n} < 1-\frac{i}{n+1}.$$

Hence,

$$a_n \leqslant a_{n+1}.$$

Furthermore, we have

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^{n} &= 1+\sum_{k=1}^{n}\frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{k-1}{n}) \\
&\leqslant 1+\sum_{k=1}^{n}\frac{1}{k!} \\
&\leqslant 1+1+\sum_{k=2}^{n}\frac{1}{k!} \\
&\leqslant 1+1+\sum_{k=2}^{n}\frac{1}{(k-1)k} \\
&\leqslant 1+1+\sum_{k=2}^{n}(\frac{1}{k-1}-\frac{1}{k}) \\
&= 1+1+1-\frac{1}{n} \\
&< 3.
\end{aligned}$$

Thus the sequence (a_n) is a monotone sequence and (a_n) is bounded. By monotone convergence theorem we conclude that (a_n) is convergent.

Note that

$$2 = a_1 \leqslant a_n \leqslant 3.$$

This implies that $2 \leqslant e \leqslant 3$. In fact $e = 2.718.....$

The form definition of the limit: If for any $\epsilon > 0$ there exits $N = N_\epsilon$ such that $n \geqslant N$ implies

$$|a_n - L| < \epsilon.$$

Then we way that L is the limit of the sequence (a_n).